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MODES OF LOSS OF STABILITY AND CRITICAL LOADS OF A THREE-LAYER SPHERICAL SHELL UNDER A UNIFORM EXTERNAL PRESSURE†

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Problems of the modes of loss of stability of a three-layer spherical shell, consisting of thin external layers and a transversely soft filler of arbitrary thickness, which is under conditions of a uniform external pressure, are considered. The two-dimensional equations of the Kirchhoff–Love theory of the moderate flexure of thin shells are used. These equations are set up for the external layers, taking account of the interaction with the filler and, in the case of the filler, using the geometrically non-linear equations of the theory of elasticity, which correspond to the introduction of the assumption that the stretching deformations are small and the shear deformation are finites, which enables the purely shear modes of loss of stability in the filler to be described correctly. An exact analytical solution is found for the problem of an initial centro-symmetric deformation of a shell, which depends linearly on the external pressure. It is shown that the three-dimensional equations for the filler, which have been linearized in the neighbourhood of this solution, can be integrated with respect to the radial coordinate, and reduce to two two-dimensional differential equations, in addition to the six equations by which the neutral equilibrium of the external layers is described. It is established that the system of eight differential equations of stability, constructed for a shell with isotropic layers, when new unknowns in the form of scalar and vortex potentials are introduced, decomposes into two unconnected systems of equations. The first of these systems has two forms of solutions by which the shear modes of loss of stability are described for the same value of the critical load. A mixed flexural mode, the realization of which is possible for certain combinations of the governing parameters of the shell for high values of the external pressure compared with the shear modes, is described by the second system.

It was shown in [1] that, under the action of a uniform external pressure, the realization of a purely shear mode of loss of stability is also possible in a three-layer ring for certain combinations of the governing parameters in addition to a composite flexural mode [2]. The beginning of this process is associated with a rotation of one of the supporting layers with respect to the other, solely because of a transverse shear deformation which is constant in a peripheral direction. A more detailed study of this problem, carried out in a geometrically non-linear formulation, showed [3] that, after crossing the shear branching point, which is located in the initial linear section of the solution concerning axisymmetric deformation, when the external pressure is increased furthers, the deformation of the ring, while remaining axisymmetric, is accompanied by a further mutual convergence of the external layers because of the increasing deformations of the transverse compression of the filler, with their simultaneous mutual rotation. The final loss of stability of the ring by virtue of the characteristic of the stress–strain state in the shear branch of the solution, which has been noted, can apparently only occur through a mixed flexural-shear mode [4, 5].

The results in [6] have proved to be important for analysing the shear modes of three-layer structures. According to these results, the stability equations, derived earlier ([7, 8], etc.) and used in [1–3], contain only secondary parametric terms of the description and development of shear modes. The principal reason for the realization of these modes when there are no subcritical transverse shear stresses lies in the occurrence in it of subcritical compressive stresses in a transverse direction and, for the correct description of these modes in the unperturbed state, it is necessary to assume that the transverse shears

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in the filler are finite. It is sufficient for this purpose to retain the non-linear terms in the tangential components of the displacements in the expression for the transverse compression of the filler.

In this connection, the problems of the modes of loss of stability of a three-layer spherical shell under a uniform external pressure studied in this paper are, unlike in the alternative approach [3], based on the use of more accurate equations which, as regards their accuracy and content, fully correspond to both the requirements mentioned above as well as to the requirements formulated earlier in [5].

1. GEOMETRICALLY NON-LINEAR EQUATIONS OF THE IMPROVE THEORY OF SPHERICAL SHELLS WITH A TRANSVERSELY SOFT FILLER OF ARBITRARY THICKNESS

Consider a closed three-layer spherical shell consisting of two load-bearing layers with thickness $2t_{(k)}$ ($k = 1$ corresponds to the lower layer and $k = 2$ to the upper layer) and a transversely soft filler of thickness $2h$. We relate the middle surface of the filler σ, which has a radius R, to a geographical system of coordinates, that is, to the angles of latitude θ ($0 \le \theta \le 2\pi$) and longitude φ ($-\pi \le \varphi \le \pi$). Assuming that the materials of the load-bearing layers and the filler are orthotropic and that the axes of orthotropism coincide with the directions of the coordinate lines of the chosen system of coordinates, we denote the elastic characteristics (the moduli of elasticity and Poisson's ratios) by $E_1^{(k)}$, $E_2^{(k)}$, $G_{12}^{(k)}$, $\nu_1^{(k)}$, $\nu_2^{(k)}$and the moduli of elasticity in the direction of the normal to the surface σ and the transverse shear moduli of the filler by E_3, G_{13}, G_{23}.

If the space of the filler is referred to a triorthogonal system of coordinates θ, φ and z, which is normally associated with the surface σ, and a dimensionless radial coordinate is introduced into the treatment instead of the transverse coordinate $z(-h \le z \le h)$, then the Lamé parameters at an arbitrary point of the filler, which is spaced at a level z from σ, will have the form $H_1 = \rho R$, $H_2 = \rho R\sin\theta$. At the same time, the following kinematic relations in the three-dimensional theory of elasticity can be written for the filler

$$2\varepsilon_{\theta z} = \frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\frac{U}{\rho}\right) + \frac{\partial W}{\rho W\partial\theta}, \quad 2\varepsilon_{\varphi z} = \frac{\partial W}{\rho R\sin\theta\partial\varphi} + \frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\frac{V}{\rho}\right)$$
$$\varepsilon_{zz} = \frac{\partial W}{R\partial\rho} + \frac{1}{2R^2}\left[\left(\frac{\partial U}{\partial\rho}\right)^2 + \left(\frac{\partial V}{\partial\rho}\right)^2\right] \tag{1.1}$$

which, while satisfying the requirements formulated earlier in [6] in relation to the possibility of a correct description of the purely shear modes of loss of stability, are simplified to the greatest extent as regards the retention of the minimum number of geometrically non-linear terms. The three-dimensional equations for the equilibrium of a transversely soft [9] filler in projections onto undeformed axes

$$\frac{\partial}{\partial\rho}(\rho^2\sigma^*_{z\theta}) + \rho\sigma_{\theta z} = 0, \quad \frac{\partial}{\partial\rho}(\rho^2\sigma^*_{z\varphi}) + \rho\sigma_{\varphi z} = 0$$
$$\frac{\partial}{\partial\rho}(\rho^2\sigma_{zz}) + \frac{\rho}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\sigma_{\theta z}) + \frac{\partial\sigma_{\varphi z}}{\partial\varphi}\right] = 0 \tag{1.2}$$

correspond to relations (1.1) which have been constructed, in which the components of the stresses $\sigma^*_{z\theta}$, $\sigma^*_{z\varphi}$, $\sigma_{\theta z}$, σ_{zz}, referred to the undeformed and deformed axes respectively, are connected by the relations

$$\sigma^*_{z\theta} = \sigma_{\theta z} + \sigma_{zz}\frac{\partial U}{R\partial\rho}, \quad \sigma^*_{z\varphi} = \sigma_{\varphi z} + \sigma_{zz}\frac{\partial V}{R\partial\rho} \tag{1.3}$$

Moreover, the Hooke's law relations

$$\sigma_{z\theta} = 2G_{13}\varepsilon_{z\theta}, \quad \sigma_{z\varphi} = 2G_{23}\varepsilon_{z\varphi}, \quad \sigma_{zz} = E_3\varepsilon_{zz} \tag{1.4}$$

hold for the stresses $\sigma_{\theta z} = \sigma_{z\theta}$, $\sigma_{\varphi z} = \sigma_{z\varphi}$, σ_{zz} within the limits of linearly-elastic deformations by virtue of the well-known equalities [9] $\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \sigma_{\theta\varphi} = 0$.

We will describe the equilibrium of the external layers accompanying their moderate flexure by the equations of the classical Kirchhoff–Love theory of shells which, taking into account the strong interaction with the filler, can be represented in projections onto the undeformed axes in the form

$$
\begin{aligned}
f_\theta^{(k)} &= \frac{\partial T_{\theta\theta}^{(k)}}{\partial\theta} + \operatorname{ctg}\theta(T_{\theta\theta}^{(k)} - T_{\varphi\varphi}^{(k)}) + \frac{\partial T_{\theta\varphi}^{(k)}}{\sin\theta\partial\varphi} + \frac{S_\theta^{(k)}}{R\rho_{(k)}\sin\theta} + R\rho_{(k)}\delta_{(k)}\sigma_{z\theta}^* = 0 \\
f_\varphi^{(k)} &= \frac{\partial T_{\theta\varphi}^{(k)}}{\partial\theta} + 2\operatorname{ctg}\theta T_{\varphi\varphi}^{(k)} + \frac{\partial T_{\varphi\varphi}^{(k)}}{\sin\theta\partial\varphi} + \frac{S_\varphi^{(k)}}{R\rho_{(k)}\sin\theta} + R\rho_{(k)}\delta_{(k)}\sigma_{z\varphi}^* = 0 \\
f_z^{(k)} &= T_{\theta\theta}^{(k)} + T_{\varphi\varphi}^{(k)} - \frac{\partial S_\theta^{(k)}}{R\rho_{(k)}\sin\theta\partial\theta} - \frac{\partial S_\varphi^{(k)}}{R\rho_{(k)}\sin^2\theta\partial\varphi} + R\rho_{(k)}\delta_{(k)}\sigma_{zz}^* + R\rho_{(k)}P_{(k)} = 0
\end{aligned}
\tag{1.5}
$$

$$k = 1,2; \quad \delta_{(1)} = 1, \quad \delta_{(2)} = -1$$

The transverse shear and normal stresses in the filler, acting on the external layers at the points of the contact surfaces and which are defined in projections onto the undeformed axes and depend on θ, φ, $\rho_{(k)}$, are denoted by $\sigma_{z\theta}^*$, $\sigma_{z\varphi}^*$, σ_{zz}^*; $P_{(k)}$ are the normal components of the external surface load.

The kinematic conditions for the joining of the external layers to the filler

$$
\begin{aligned}
&w^{(k)} = W(\theta, \varphi, \rho_{(k)}), \quad u^{(k)} + \overset{0}{t}_{(k)}\delta_{(k)}\omega_\theta^{(k)} = U(\theta, \varphi, \rho_{(k)}) \\
&v^{(k)} + \overset{0}{t}_{(k)}\delta_{(k)}\omega_\varphi^{(k)} = V(\theta, \varphi, \rho_{(k)}); \quad k = 1,2
\end{aligned}
\tag{1.6}
$$

have to be combined with the equilibrium equations presented above. The notation

$$\overset{0}{t}_{(k)} = \frac{t_{(k)}}{R}, \quad \rho_{(k)} = 1 - \delta_{(k)}h_0, \quad h_0 = \frac{h}{R}$$

$$\omega_\theta^{(k)} = \frac{\partial w^{(k)}}{\partial\theta} - u^{(k)}, \quad \omega_\varphi^{(k)} = \frac{\partial w^{(k)}}{\sin\theta\partial\varphi} - v^{(k)}$$

has been introduced into Eqs (1.6).

For moderate flexure of a shell, the shearing forces in the load-bearing layers in relation (1.5) are connected in the quadratic approximation, in terms of the internal shear forces $T_{\theta\theta}^{(k)}$, $T_{\varphi\varphi}^{(k)}$, $T_{\theta\varphi}^{(k)}$ and the moments $M_{\theta\theta}^{(k)}$, $M_{\varphi\varphi}^{(k)}$, $M_{\theta\varphi}^{(k)}$ by the relations

$$
\begin{aligned}
S_\theta^{(k)} &= \sin\theta\frac{\partial M_{\theta\theta}^{(k)}}{\partial\theta} + \cos\theta(M_{\theta\theta}^{(k)} - M_{\varphi\varphi}^{(k)}) + \frac{\partial M_{\theta\varphi}^{(k)}}{\partial\varphi} + T_{\theta\theta}^{(k)}\omega_\theta^{(k)}\sin\theta \\
S_\varphi^{(k)} &= \sin\theta\frac{\partial M_{\theta\varphi}^{(k)}}{\partial\theta} + 2\cos\theta M_{\theta\varphi}^{(k)} + \frac{\partial M_{\varphi\varphi}^{(k)}}{\partial\varphi} + T_{\varphi\varphi}^{(k)}\omega_\varphi^{(k)}\sin\theta
\end{aligned}
\tag{1.7}
$$

and the elasticity relations

$$
\begin{aligned}
T_{\theta\theta}^{(k)} &= B_1^{(k)}\left[\frac{\partial u^{(k)}}{\partial\theta} + \frac{\omega_\theta^{(k)2}}{2} + \nu_1^{(k)}\left(w^{(k)} + \frac{\partial v^{(k)}}{\sin\theta\partial\varphi} + \operatorname{ctg}\theta u^{(k)} + \frac{\omega_\varphi^{(k)2}}{2}\right)\right] \\
T_{\theta\varphi}^{(k)} &= B_{12}^{(k)}\left[\frac{\partial u^{(k)}}{\sin\theta\partial\varphi} + \frac{\partial v^{(k)}}{\partial\theta} - \operatorname{ctg}\theta v^{(k)} + \omega_\theta^{(k)}\omega_\varphi^{(k)}\right] \\
T_{\varphi\varphi}^{(k)} &= B_2^{(k)}\left[w^{(k)} + \frac{\partial v^{(k)}}{\sin\theta\partial\varphi} + \operatorname{ctg}\theta u^{(k)} + \frac{\omega_\varphi^{(k)2}}{2} + \nu_2^{(k)}\left(\frac{\partial u^{(k)}}{\partial\theta} + \frac{\omega_\theta^{(k)2}}{2}\right)\right]
\end{aligned}
\tag{1.8}
$$

$$M_{\theta\theta}^{(k)} = -D_1^{(k)}\left[\frac{\partial^2 w^{(k)}}{\partial\theta^2} - \frac{\partial u^{(k)}}{\partial\theta} + \nu_1^{(k)}\left(\frac{\partial^2 w^{(k)}}{\sin^2\theta\partial\varphi^2} + \operatorname{ctg}\theta\frac{\partial w^{(k)}}{\partial\theta} - \frac{\partial \upsilon^{(k)}}{\sin\theta\partial\varphi} - \operatorname{ctg}\theta u^{(k)}\right)\right]$$

$$M_{\theta\varphi}^{(k)} = -D_{12}^{(k)}\left[2\frac{\partial^2 w^{(k)}}{\sin\theta\partial\theta\partial\varphi} - 2\operatorname{ctg}\theta\frac{\partial w^{(k)}}{\partial\varphi} - \frac{\partial u^{(k)}}{\sin\theta\partial\varphi} - \frac{\partial \upsilon^{(k)}}{\partial\theta} + \operatorname{ctg}\theta\,\upsilon^{(k)}\right] \quad (1.9)$$

$$M_{\varphi\varphi}^{(k)} = -D_2^{(k)}\left[\frac{\partial^2 w^{(k)}}{\sin^2\theta\partial\varphi^2} + \operatorname{ctg}\theta\frac{\partial w^{(k)}}{\partial\theta} - \frac{\partial \upsilon^{(k)}}{\sin\theta\partial\varphi} - \operatorname{ctg}\theta u^{(k)} + \nu_2^{(k)}\left(\frac{\partial^2 w^{(k)}}{\partial\theta^2} - \frac{\partial u^{(k)}}{\partial\theta}\right)\right]$$

hold within the limits of the elastic deformations.

Here

$$B_j^{(k)} = \frac{2E_j^{(k)}t_{(k)}^0}{\rho_{(k)}(1-\nu_1^{(k)}\nu_2^{(k)})}, \quad D_j^{(k)} = \frac{B_j^{(k)}t_{(k)}^2}{3R\rho_{(k)}}; \quad j = 1, 2$$

$$B_{12}^{(k)} = \frac{2t_{(k)}^0 G_{12}^{(k)}}{\rho_{(k)}}, \quad D_{12}^{(k)} = \frac{B_{12}^{(k)}t_{(k)}^2}{3R\rho_{(k)}} \quad (1.10)$$

are the stiffness characteristics of the load-bearing layers.

After introducing the dimensionless governing parameters $\gamma_j^{(k)} = B_{12}^{(k)}/B_j^{(k)}$ and the differential operators

$$L_{11}^{(k)} = \frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} - \nu_1^{(k)} - \frac{\nu_1^{(k)}}{\nu_2^{(k)}}\operatorname{ctg}^2\theta + \frac{\gamma_1^{(k)}\partial^2}{\sin^2\theta\partial\varphi^2}$$

$$L_{12}^{(k)} = \frac{\nu_2^{(k)}}{\nu_1^{(k)}}\frac{\partial}{\sin\theta\partial\varphi}\left[(\nu_2^{(k)} + \gamma_2^{(k)})\frac{\partial}{\partial\theta} - (1 + \gamma_2^{(k)})\operatorname{ctg}\theta\right]$$

$$L_{21}^{(k)} = \frac{\partial}{\sin\theta\partial\varphi}\left[(\nu_2^{(k)} + \gamma_2^{(k)})\frac{\partial}{\partial\theta} + (1 + \gamma_2^{(k)})\operatorname{ctg}\theta\right] \quad (1.11)$$

$$L_{22}^{(k)} = \gamma_2^{(k)}\left(\frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + 1 - \operatorname{ctg}^2\theta\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

when (1.9) is used, the shearing forces (1.7) will be expressed in terms of the displacements $u^{(k)}$, $\upsilon^{(k)}$, $w^{(k)}$ of points of the middle surfaces of load-bearing layers by the formulae

$$S_\theta^{(k)} = D_1^{(k)}\sin\theta\left\{L_{11}^{(k)}(u^{(k)}) + L_{12}^{(k)}(\upsilon^{(k)}) - \left(L_{11}^{(k)} - \frac{\gamma_1^{(k)}}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{\partial w^{(k)}}{\partial\theta}\right) - \right.$$

$$\left. - \frac{\partial^2}{\sin^2\theta\partial\varphi^2}\left[(\nu_1^{(k)} + 2\gamma_1^{(k)})\left(\frac{\partial}{\partial\theta} - \operatorname{ctg}\theta\right) - \frac{\nu_1^{(k)}}{\nu_2^{(k)}}\operatorname{ctg}\theta\right]w^{(k)}\right\} + T_{\theta\theta}^{(k)}\omega_\theta^{(k)}\sin\theta$$

$$S_\varphi^{(k)} = D_2^{(k)}\sin\theta\left\{L_{21}^{(k)}(u^{(k)}) + L_{22}^{(k)}(\upsilon^{(k)}) - \frac{\partial}{\sin\theta\partial\varphi}\left[(\nu_2^{(k)} + 2\gamma_2^{(k)})\frac{\partial^2}{\partial\theta^2} + \right.\right. \quad (1.12)$$

$$\left.\left. + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + 2\gamma_2^{(k)} + \frac{\partial^2}{\sin^2\theta\partial\varphi^2}\right]w^{(k)}\right\} + T_{\varphi\varphi}^{(k)}\omega_\varphi^{(k)}\sin\theta$$

2. CENTRO-SYMMETRIC DEFORMATION OF A SHELL

If the external pressure p is uniformly distributed, the stress–strain state of the shell under investigation, which depends linearly on the external load p, will be described, for centro-symmetric deformation, when

$$u^{0(k)} = v^{0(k)} = 0,\quad \sigma^*_{z\theta} = \overset{0}{\sigma}_{z\theta} = 0,\quad \sigma^*_{z\varphi} = \overset{0}{\sigma}_{z\varphi} = 0,\quad \sigma^*_{zz} = \sigma_{zz} = \overset{0}{\sigma}_{zz}$$

by the equilibrium equations

$$\frac{d}{d\rho}(\rho^2\overset{0}{\sigma}_{zz}) = 0,\quad f^{(k)}_{zz} = T^{0(k)}_{\theta\theta} + T^{0(k)}_{\varphi\varphi} - \overset{0}{\sigma}_{zz}R\rho_{(k)}\delta_{(k)} + P_{(k)} = 0,\quad k = 1,2 \tag{2.1}$$

Here and henceforth, the parameters of this stress–strain state are labelled with additional zero superscripts.

Combining the physical relations

$$T^{0(k)}_{\theta\theta} = B^{(k)}_1(1+\nu^{(k)}_1)w^{0(k)},\quad T^{0(k)}_{\varphi\varphi} = B^{(k)}_2(1+\nu^{(k)}_2)w^{0(k)},\quad \overset{0}{\sigma}_{zz} = E_3\frac{dW^0}{Rd\rho} \tag{2.2}$$

with Eqs (2.1), we successively find the integrals

$$\overset{0}{\sigma}_{zz} = \frac{q_0}{\rho^2},\quad W^0 = w_0 - \frac{Rq_0}{\rho E_3} \tag{2.3}$$

where q_0 and w_0 are constants, which are determined from the contact condition

$$w^{0(k)} = W^0(\rho_{(k)}),\quad k = 1,2$$

By satisfying these conditions we can determine the radial stress

$$\overset{0}{\sigma}_{zz} = \frac{E_3\rho_{(1)}\rho_{(2)}}{2R\rho^2h_0}(w^{0(2)} - w^{0(1)}) \tag{2.4}$$

Using relations (2.2) and (2.4), a system of algebraic equations follows from Eqs (2.1), which are written in terms of the deflection $w^{0(1)}$ and $w^{0(2)}$

$$(1+\chi_{(k)})w^{0(k)} - \chi_{(k)}w^{0(3-k)} = -\delta_{2k}\frac{p\rho_{(k)}}{\mu^{(k)}_1B^{(k)}_2},\quad \delta_{21} = 0,\quad \delta_{22} = 1,\quad k = 1,2 \tag{2.5}$$

where

$$\mu^{(k)}_1 = 1 + 2\nu^{(k)}_2 + \frac{\nu^{(k)}_2}{\nu^{(k)}_1},\quad \chi_{(k)} = \frac{E_3\rho_{(1)}\rho_{(2)}(1-\nu^{(k)}_1\nu^{(k)}_2)}{4h_0\overset{0}{t}_{(k)}E^{(k)}_2\mu^{(k)}_1}$$

The deflections of the load-bearing layers

$$w^{0(k)} = -\frac{pR\rho_{(2)}(k-1+\chi_{(1)})}{\mu^{(2)}_1B^{(2)}_2(1+\chi_{(1)}+\chi_{(2)})} \tag{2.6}$$

are determined from system (2.5), and the forces in these layers are found from relations (2.2).

It follows from expressions (2.2) and (2.6) that, in the general case, the forces $T^{0(k)}_{\theta\theta}$ are different from the forces $T^{0(k)}_{\varphi\varphi}$ for the orthotropic materials of the load-bearing layers. It can be shown that they are equal to one another when the condition $(1 + \nu^{(k)}_1)\nu^{(2)}_1 = (1 + \nu^{(k)}_2)\nu^{(2)}_2$ are satisfied. In the special case when the materials of the load-bearing layers are isotropic, that is, when $\nu^{(k)}_1 = \nu^{(k)}_2 = \nu^{(k)}$, $E^{(k)}_1 = E^{(k)}_2 = E^{(k)}$, we have

$$T_{\theta\theta}^{0(1)} = T_{\varphi\varphi}^{0(1)} = -\frac{pR\rho_{(2)}\chi_{(1)}^{*}(1+\nu^{(1)})}{2(1+\chi_{(1)}^{*}+\chi_{(2)}^{*})(1+\nu^{(2)})}, \quad T_{\theta\theta}^{0(2)} = T_{\varphi\varphi}^{0(2)} = -\frac{pR\rho_{(2)}(1+\chi_{(1)}^{*})}{2(1+\chi_{(1)}^{*}+\chi_{(2)}^{*})} \tag{2.7}$$

where

$$\chi_{(k)}^{*} = \frac{E_3\rho_{(1)}\rho_{(2)}(1-\nu^{(k)})}{8h_0\overset{0}{t}_{(k)}E_2^{(k)}}, \quad k = 1,2$$

When there is no filler ($E_3 = 0, \chi_{(1)}^{*} = \chi_{(2)}^{*} = 0$), the well-known results

$$T_{\theta\theta}^{0(1)} = T_{\varphi\varphi}^{0(1)} = 0, \quad T_{\theta\theta}^{0(2)} = T_{\varphi\varphi}^{0(2)} = -pR\rho_{(2)}/2$$

follow from Eqs (2.7).

According to expression (2.6), for the radial stress (2.4), we have

$$\sigma_{zz}^{0} = -\frac{p\rho_{(2)}^{2}\chi_2}{(1+\chi_{(1)}+\chi_{(2)})\rho^{2}} \tag{2.8}$$

3. THE LINEARIZED STABILITY EQUATIONS

To determine the values of p, on reaching which branching of the solution of the composite system of non-linear equilibrium equations is possible, we will linearize them in the neighbourhood of the solution (2.2), (2.6), (2.8).

The linearized equilibrium equations for the filler and their reduction to two-dimensional equations. If we assume that, prior to the loss of stability, the shell is stressed but not deformed, then, by virtue of the relations $U^0 = V^0 = 0$, $\sigma_{\theta z}^{0} = \sigma_{\varphi z}^{0} = 0$, the neutral equilibrium equations for the filler take the form

$$\frac{\partial}{\partial\rho}(\rho^2\sigma_{z\theta}^{*}) + \rho\sigma_{\theta z} = 0, \quad \frac{\partial}{\partial\rho}(\rho^2\sigma_{z\varphi}^{*}) + \rho\sigma_{\varphi z} = 0$$
$$\frac{\partial}{\partial\rho}(\rho^2\sigma_{zz}) + \frac{\rho}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\sigma_{\theta z}) + \frac{\partial\sigma_{\varphi z}}{\partial\varphi}\right] = 0 \tag{3.1}$$

Here, unlike relations (1.3).

$$\sigma_{z\theta}^{*} = \sigma_{\theta z} + \sigma_{zz}^{0}\frac{\partial U}{R\partial\rho} = G_{13}\left[\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\frac{U}{\rho}\right) + \frac{\partial W}{\rho R\partial\theta}\right] + \sigma_{zz}^{0}\frac{\partial U}{R\partial\rho}$$
$$\sigma_{\varphi z}^{*} = \sigma_{\varphi z} + \sigma_{zz}^{0}\frac{\partial V}{R\partial\rho} = G_{23}\left[\frac{\partial W}{\rho R\sin\theta\partial\varphi} + \frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\frac{V}{\rho}\right)\right] + \sigma_{zz}^{0}\frac{\partial V}{R\partial\rho} \tag{3.2}$$

$$\sigma_{zz} = E_3\frac{\partial W}{R\partial\rho} \tag{3.3}$$

and σ_{zz}^{0} is calculated using formula (2.8).

The functions

$$\sigma_{z\theta} = \frac{q_1}{\rho^3} - \frac{\overset{0}{q_3}}{\rho R}\frac{\partial}{\partial\rho}\left(\frac{U}{\rho}\right), \quad \sigma_{z\varphi} = \frac{q_2}{\rho^3} - \frac{\overset{0}{q_3}}{\rho R}\frac{\partial}{\partial\rho}\left(\frac{V}{R}\right) \tag{3.4}$$

$$\sigma_{zz} = \frac{q_3}{\rho^2} + \frac{q}{\rho^3} + \overset{0}{q_3}\frac{S}{\rho^3} \tag{3.5}$$

in which $q_i = q_i(\theta, \varphi)$ are functions of integration and

$$\overset{0}{q}_3 = -\frac{p\rho_{(2)}^2\chi_{(2)}}{1+\chi_{(1)}+\chi_{(2)}}, \quad q = \frac{\partial q_1}{\partial\theta} + \mathrm{ctg}\theta q_1 + \frac{\partial q_2}{\sin\theta\partial\varphi}$$
$$S = \frac{1}{R}\left(\frac{\partial U}{\partial\theta} + \mathrm{ctg}U + \frac{\partial V}{\sin\theta\partial\varphi}\right) \tag{3.6}$$

are the integrals of Eqs (3.1).

The radial displacement in the filler

$$W = w_0(\theta, \varphi) - \frac{R}{E_3}\left(\frac{q_3}{\rho} + \frac{q}{2\rho^2} - \overset{0}{q}_3 J\right), \quad J = \int\frac{Sd\rho}{\rho^3} \tag{3.7}$$

where w_0 is still an arbitrary function, is determined during the subsequent integration of Eq (3.3) and (3.5).

By satisfying the kinematic contact conditions along the normal to the surface σ from (1.6) we can determine the integration functions q_3 and w_0, and, substituting them into expression (3.7), we obtain the following formula for determining of the buckling in the filler

$$W = \frac{\rho_{(2)}(\tilde{z}+h_0)w^{(2)} - \rho_{(1)}(\tilde{z}-h_0)w^{(1)}}{2h_0\rho} - \frac{qR(\tilde{z}^2-h_0^2)}{2E_3\rho_{(1)}\rho_{(2)}\rho^2} +$$
$$+\overset{0}{q}_3 R\frac{\rho_{(1)}(\tilde{z}-h_0)J_{(1)} - \rho_{(2)}(\tilde{z}+h_0)J_{(2)} + 2h_0\rho J}{2h_0E_3\rho}; \quad \tilde{z} = \frac{z}{R}, \quad J_{(k)} = J(\theta, \varphi, \rho_{(k)}) \tag{3.8}$$

When $S\rho^{-3} \leq R$, the term containing the factor $\overset{0}{q}_3$ can be neglected, with an accuracy $O(h_0^2)$, compared with the first term. Then, for W, we will have the simplified formula

$$W \approx \frac{\rho_{(2)}(\tilde{z}+h_0)w^{(2)} - \rho_{(1)}(\tilde{z}-h_0)w^{(1)}}{2h_0\rho} - \frac{qE(\tilde{z}^2-h_0^2)}{2E_3\rho_{(1)}\rho_{(2)}\rho^2} \tag{3.9}$$

and σ_{zz} will be calculated using the formula

$$\sigma_{zz} \approx \frac{E_3\rho_{(1)}\rho_{(2)}(w^{(2)}-w^{(1)})}{2h\rho^2} + q\left(\frac{1}{\rho^3} - \frac{1}{\rho_{(1)}\rho_{(2)}}\right) \tag{3.10}$$

In order to determine the tangential displacements in the filler, we make use of relations (1.4) and (3.2), from which the differential equations

$$\left(G_{13} + \frac{\overset{0}{q}_3}{\rho^2}\right)\frac{\partial}{\partial\rho}\left(\frac{U}{\rho}\right) = \frac{q_1R}{\rho^4} - \frac{G_{13}\partial W}{\rho^2\,\partial\theta}$$
$$\left(G_{23} + \frac{\overset{0}{q}_3}{\rho^2}\right)\frac{\partial}{\partial\rho}\left(\frac{V}{\rho}\right) = \frac{q_2R}{\rho^4} - \frac{G_{23}}{\rho^2\sin\theta}\frac{\partial W}{\partial\varphi} \tag{3.11}$$

follow. Integration of these equations is difficult because of the existence of variable coefficients in front of the derivatives with respect to the coordinate ρ. We shall therefore determine their solution in the neighbourhood

$$\frac{\overset{0}{q}_3}{\rho^2} = \overset{0}{\sigma}_{zz} \approx -\frac{p\chi_{(2)}}{1+\chi_{(1)}+\chi_{(2)}} = \tilde{q} \tag{3.12}$$

It can be represented in the form

$$
\begin{aligned}
\frac{U}{\rho} &= U_0(\theta,\varphi)-\frac{q_1 R}{3G_{13}^*\rho^3}-\frac{G_{13}}{G_{13}^*}\frac{\partial}{\partial\theta}\left(\int\frac{Wd\rho}{\rho^2}\right)\\
\frac{V}{\rho} &= V_0(\theta,\varphi)-\frac{q_2 R}{3G_{23}^*\rho^3}-\frac{G_{23}}{G_{23}^*}\frac{\partial}{\sin\theta\partial\varphi}\left(\int\frac{Wd\rho}{\rho^2}\right)
\end{aligned}
\tag{3.13}
$$

where $G_{13}^* = G_{13} + \tilde{q}$, $G_{23}^* = G_{23} + \tilde{q}$. On introducing the expression for the radial displacement (3.9) into equalities (3.3) and satisfying the kinetic matching conditions with respect to the tangential displacements from (1.6), we arrive, after some reduction, at the two-dimensional differential equations

$$
\begin{aligned}
\mu_1 &= \frac{\omega_\theta^{(2)}}{\rho_{(2)}}-\frac{\omega_\theta^{(1)}}{\rho_{(1)}}+\frac{\partial w_1}{\partial\theta}+f_1-g_1\frac{\partial q}{\partial\theta} = 0\\
\mu_2 &= \frac{\omega_\varphi^{(2)}}{\rho_{(2)}}-\frac{\omega_\varphi^{(1)}}{\rho_{(1)}}+\frac{\partial w_2}{\sin\theta\partial\varphi}+f_2-g_2\frac{\partial q}{\sin\theta\partial\varphi} = 0
\end{aligned}
\tag{3.14}
$$

in which

$$
w_i = \frac{\rho_{(2)}w^{(1)}-\rho_{(1)}w^{(2)}-h_i^*(w^{(1)}+w^{(2)})}{\rho_{(1)}\rho_{(2)}};\quad h_i^* = \frac{h_0 G_{i3}}{G_{i3}^*}
$$

$$
f_i = \frac{q_i R f}{G_{i3}^*},\quad g_i = \frac{2h_0^3 G_{i3}R}{3E_3 G_{i3}^*\rho_{(1)}^3\rho_{(2)}^3};\quad i = 1, 2;\quad f = 2h_0(3+h_0^2)/(3p_1^3p_2^3)
$$

Linearization of the stability equation for the load-bearing layers. Linearizing Eqs (1.5), (1.8) and (1.12) in the neighbourhood of the solutions (2.6), (2.7), (2.9), we arrive at the neutral equilibrium equations of the external layers which can be represented in terms of the displacements in the form

$$
\begin{aligned}
f_\theta^{(k)} &= L_{11}^{(k)}(u^{(k)})+L_{12}^{(k)}(v^{(k)})+\left[(1+\nu_1^{(k)})\frac{\partial}{\partial\theta}+\left(1-\frac{\nu_1^{(k)}}{\nu_2^{(k)}}\right)\mathrm{ctg}\,\theta\right]w^{(k)}-\\
&-C_{(k)}^2\left[\frac{\partial^2}{\partial\theta^2}+\mathrm{ctg}\,\theta\frac{\partial}{\partial\theta}-\nu_1^{(k)}-\frac{\nu_1^{(k)}}{\nu_2^{(k)}}\mathrm{ctg}^2\theta\right]\frac{\partial w^{(k)}}{\partial\theta}-\\
&-C_{(k)}^2\frac{\partial^2}{\sin^2\theta\partial\varphi^2}\left[(\nu_1^{(k)}+2\gamma_1^{(k)})\left(\frac{\partial}{\partial\theta}-\mathrm{ctg}\,\theta\right)-\frac{\nu_1^{(k)}}{\nu_2^{(k)}}\mathrm{ctg}\,\theta\right]w^{(k)}+\\
&+\frac{T_{\theta\theta}^{0(k)}\omega_\theta^{(k)}}{R\rho_{(k)}B_1^{(k)}}+\frac{Rq_1^*(\rho_{(k)})\delta_{(k)}}{\rho_{(k)}^2B_1^{(k)}} = 0\\
f_\varphi^{(k)} &= L_{21}^{(k)}(u^{(k)})+L_{22}^{(k)}(v^{(k)})+(1+\nu_2^{(k)})\frac{\partial w^{(k)}}{\sin\theta\partial\varphi}-\\
&-C_{(k)}^2\frac{\partial}{\sin\theta\partial\varphi}\left[(\nu_2^{(k)}+2\gamma_2^{(k)})\frac{\partial^2}{\partial\theta^2}+2\gamma_2^{(k)}+\mathrm{ctg}\,\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\sin^2\theta\partial\varphi^2}\right]w^{(k)}+\\
&+\frac{T_{\varphi\varphi}^{0(k)}\omega_\varphi^{(k)}}{R\rho_{(k)}B_2^{(k)}}+\frac{Rq_2^*(\rho_{(k)})\delta_{(k)}}{\rho_{(k)}^2B_2^{(k)}} = 0
\end{aligned}
\tag{3.15}
$$

$$f_z^{(k)} = \left(\nu_2^{(k)} + \frac{\nu_2^{(k)}}{\nu_1^{(k)}}\right)\frac{\partial u^{(k)}}{\partial\theta} + (1+\nu_2^{(k)})\operatorname{ctg}\theta u^{(k)} + (1+\nu_2^{(k)})\frac{\partial v^{(k)}}{\sin\theta\partial\varphi} + \mu_1^{(k)} w^{(k)} -$$

$$-C_{(k)}^2\left\{\frac{\nu_2^{(k)}}{\nu_1^{(k)}}\left(\frac{\partial}{\partial\theta} + 2\operatorname{ctg}\theta\right)\frac{\partial^2 u^{(k)}}{\partial\theta^2} + \frac{\partial^2}{\sin^2\theta\partial\varphi^2}\left[(\nu_2^{(k)} + 2\gamma_2^{(k)})\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\right]u^{(k)} +\right.$$

$$\left.+\frac{\partial}{\sin\theta\partial\varphi}\left[(\nu_2^{(k)} + 2\gamma_2^{(k)})\frac{\partial^2}{\partial\theta^2} - \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\sin^2\theta\partial\varphi^2}\right]v^{(k)}\right\} +$$

$$+C_{(k)}^2\left\{\left[\frac{\nu_2^{(k)}}{\nu_1^{(k)}}\left(\frac{\partial^3}{\partial\theta^3} + 2\operatorname{ctg}\theta\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\theta}\right) - (\nu_2^{(k)} + \operatorname{ctg}^2\theta)\left(\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\right)\right]\frac{\partial w^{(k)}}{\partial\theta} +\right.$$

$$\left.+\frac{\partial^2}{\sin^2\theta\partial\varphi^2}\left[2(\nu_2^{(k)} + 2\gamma_2^{(k)})\left(\frac{\partial^2}{\partial\theta^2} - \operatorname{ctg}\theta\frac{\partial}{\partial\theta}\right) + (1+\nu_2^{(k)})(1+2\operatorname{ctg}^2\theta) + \frac{\partial^2}{\sin^2\theta\partial\varphi^2}\right]w^{(k)}\right\} +$$

$$+\delta_{(k)}\frac{E_3\rho_{(1)}\rho_{(2)}(w^{(1)} - w^{(2)})}{2h_0\rho_{(k)}B_2^{(k)}} - \frac{1}{R\rho_{(k)}B_2^{(k)}}\left[T_{\theta\theta}^{0(k)}\left(\frac{\partial\omega_\theta^{(k)}}{\partial\theta} + \omega_\theta^{(k)}\operatorname{ctg}\theta\right) +\right.$$

$$\left.+T_{\varphi\varphi}^{0(k)}\frac{\partial\omega_\varphi^{(k)}}{\sin\theta\partial\varphi}\right] - \frac{qRH_{(k)}}{\rho_{(k)}^2 B_2^{(k)}} = 0; \quad k = 1, 2$$

where

$$q_1^*(\rho_{(k)}) = q_1 + \frac{\overset{0}{q_3}U(\theta,\varphi,\rho_{(k)})}{R}, \quad q_2^*(\rho_{(k)}) = q_2 + \frac{\overset{0}{q_3}V(\theta,\varphi,\rho_{(k)})}{R}$$
$$C_{(k)}^2 = \frac{t_{(k)}^{02}}{3\rho_{(k)}^2}, \quad H_{(k)} = \frac{\overset{0}{t}_{(k)}}{\rho_{(k)}} + \frac{h_0\rho_{(k)}^2\delta_{(k)}}{\rho_{(1)}\rho_{(2)}} \tag{3.16}$$

Hence, the stability equations for the shell being considered consist of Eqs (3.14) and (3.15) in which the two-dimensional functions $u^{(k)}$, $v^{(k)}$, $w^{(k)}$, q_1 and q_2 are unknowns and the functions $U(\theta, \varphi, \rho_{(k)})$ and $V(\theta, \varphi, \rho_{(k)})$ occurring in (3.16) are determined for the conditions for contact of the layers with respect to the tangential displacements.

In the subsequent investigations, it is best to take the functions $w^{(k)}$, $\omega_\theta^{(k)}$, $\omega_\varphi^{(k)}$, q_1 and q_2 as unknowns, by expressing the tangential displacements in the load-bearing layers in terms of the unknowns which have been introduced according to the relations

$$u^{(k)} = \frac{\partial w^{(k)}}{\partial\theta} - \omega_\theta^{(k)}, \quad v^{(k)} = \frac{\partial w^{(k)}}{\sin\theta\partial\varphi} - \omega_\varphi^{(k)}; \quad k = 1, 2 \tag{3.17}$$

Using relations (3.17), Eqs (3.15) can be represented with an accuracy $O(C_{(k)}^2)$ in the form

$$f_\theta^{(k)} = \left[\frac{\partial}{\partial\theta}(\nabla^2 + 2) + \left(1 - \frac{\nu_1^{(k)}}{\nu_2^{(k)}}\right)\left(\nabla^2 - \frac{\partial^2}{\partial\theta^2} + 1\right) -\right.$$

$$\left.-(1 - \nu_1^{(k)} - 2\gamma_1^{(k)})\frac{\partial^2}{\sin^2\theta\partial\varphi^2}\left(\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\right)\right]w^{(k)} -$$

$$-L_{11}^{(k)}(\omega_\theta^{(k)}) - L_{12}^{(k)}(\omega_\varphi^{(k)}) + \frac{T_{\theta\theta}^{0(k)}\omega_\theta^{(k)}}{R\rho_{(k)}B_1^{(k)}} + \frac{Rq_1^*(\rho_{(k)})\delta_{(k)}}{\rho_{(k)}^2 B_1^{(k)}} = 0$$

$$f_\varphi^{(k)} = \frac{\partial}{\sin\theta\partial\varphi}\left[\nabla^2 + 2 - (1 - \nu_2^{(k)} - 2\gamma_2^{(k)})\left(\frac{\partial^2}{\partial\theta^2} + 1\right)\right]w^{(k)} -$$

$$- L_{21}^{(k)}(\omega_\theta^{(k)}) - L_{22}^{(k)}(\omega_\varphi^{(k)}) + \frac{T_{\varphi\varphi}^{0(k)}\omega_\varphi^{(k)}}{R\rho_{(k)}B_2^{(k)}} + \frac{Rq_2^*(\rho_{(k)})\delta_{(k)}}{\rho_{(k)}^2B_2^{(k)}} = 0$$

$$f_z^{(k)} = (1 + \nu_2^{(k)})\left[\nabla^2 + 2 - \left(1 - \frac{\nu_2^{(k)}}{\nu_1^{(k)}}\right)\left(\frac{\partial^2}{\partial\theta^2} + 1\right)\right]w^{(k)} + \left(1 - \frac{\nu_2^{(k)}}{\nu_1^{(k)}}\right)\frac{\partial\omega_\theta^{(k)}}{\partial\theta} - \tag{3.18}$$

$$- (1 + \nu_2^{(k)})\left(\frac{\partial\omega_\theta^{(k)}}{\partial\theta} + \operatorname{ctg}\theta\,\omega_\theta^{(k)} + \frac{\partial\omega_\varphi^{(k)}}{\sin\theta\partial\varphi}\right) + C_{(k)}^2\left\{\frac{\nu_2^{(k)}}{\nu_1^{(k)}}\left(\frac{\partial^3}{\partial\theta^3} + 2\operatorname{ctg}\theta\frac{\partial^2}{\partial\theta^2}\right) +\right.$$

$$\left. + \frac{\partial^2}{\sin^2\theta\partial\varphi^2}\left[(\nu_2^{(k)} + 2\gamma_2^{(k)})\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\right]\right\}\omega_\theta^{(k)} +$$

$$+ C_{(k)}^2\frac{\partial}{\sin\theta\partial\varphi}\left[(\nu_2^{(k)} + 2\gamma_2^{(k)})\frac{\partial^2}{\partial\theta^2} - \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\sin^2\theta\partial\varphi^2}\right]\omega_\varphi^{(k)} +$$

$$+ \delta_{(k)}\frac{E_3\rho_{(1)}\rho_{(2)}(w^{(1)} - w^{(2)})}{2h_0\rho_{(k)}B_2^{(k)}} -$$

$$- \frac{1}{R\rho_{(k)}B_2^{(k)}}\left[T_{\theta\theta}^{0(k)}\left(\frac{\partial\omega_\theta^{(k)}}{\partial\theta} + \operatorname{ctg}\theta\,\omega_\theta^{(k)}\right) + T_{\varphi\varphi}^{0(k)}\frac{\partial\omega_\varphi^{(k)}}{\sin\theta\partial\varphi}\right] - \frac{qRH_{(k)}}{\rho_{(k)}^2B_2^{(k)}} = 0$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\sin^2\theta\partial\varphi^2}$$

$$q_1^*(\rho_{(k)}) = q_1 + \frac{\overset{0}{q}_3}{R}\left[\frac{\partial w^{(k)}}{\partial\theta} - (1 - \overset{0}{t}_{(k)}\delta_{(k)})\omega_\theta^{(k)}\right] \tag{3.19}$$

$$q_2^*(\rho_{(k)}) = q_2 + \frac{\overset{0}{q}_3}{R}\left[\frac{\partial w^{(k)}}{R\sin\partial\theta} - (1 - \overset{0}{t}_{(k)}\delta_{(k)})\omega_\varphi^{(k)}\right]$$

and, in relations (3.19), the terms proportional to $\overset{0}{q}_3$ can be neglected with an accuracy of $1 + U(\theta, \varphi, \rho_{(k)})/R \approx 1$, as can be shown using formulae (2.8) and (2.9).

Investigation of the composite homogeneous system of differential equations (3.14), (3.18) shows that separation of the variables with respect to the angle θ and φ does not occur in them. Such a separation of variables is only possible for a shell with isotropic load-bearing layers an isotropic filler.

4. MODES OF LOSS OF STABILITY AND CRITICAL LOADS FOR AN ISOTROPIC THREE-LAYER SPHERICAL SHELL

If the shell is isotropic, that is, the equalities

$$E_1^{(k)} = E_2^{(k)} = E^{(k)}, \quad \nu_1^{(k)} = \nu_2^{(k)} = \nu^{(k)}(B_1^{(k)} = B_2^{(k)} = B^{(k)}), \quad G_{13} = G_{23} = G_3 \tag{4.1}$$

hold, then, on introducing dimensionless parameters, the required functions and the external load

$$K_{(k)} = \frac{3G_3^*\rho_{(1)}^2\rho_{(2)}^2}{2h_0\rho_{(k)}^2(3+h_0^2)B^{(k)}}, \quad K = \frac{G_3^* h_0^2}{E_3\rho_{(1)}\rho_{(2)}(3+h_0^2)}, \quad \varphi_{(k)} = \frac{E_3\rho_{(1)}\rho_{(2)}}{2h_0\rho_{(k)}B^{(k)}}, \quad \tilde{q}_k = \frac{fRq_k}{G_3^*}$$
$$\tilde{p}_{(1)} = \frac{p\chi_{(1)}^*(1+\nu^{(1)})}{2(1+\chi_{(1)}^*+\chi_{(2)}^*)(1+\nu^{(2)})B^{(2)}}, \quad \tilde{p}_2 = \frac{p(1+\chi_{(1)}^*)}{2(1+\chi_{(1)}^*+\chi_{(2)}^*)B^{(2)}}, \quad h_0^* = \frac{h_0G_3}{G_3^*} \tag{4.2}$$

where $G_3^* = G_3 + \tilde{q}$, the stability equations (3.14) and (3.18), neglecting the terms in relations (3.19) which are proportional to q_3^0, take the form

$$f_\theta^{(k)} = \frac{\partial}{\partial\theta}[(\nabla^2+2)w^{(k)}] - \tilde{L}_{11}^{(k)}(\omega_\theta^{(k)}) - \tilde{L}_{12}^{(k)}(\omega_\varphi^{(k)}) - \tilde{p}_{(k)}\omega_\theta^{(k)} + K_{(k)}\delta_{(k)}\tilde{q}_1 = 0$$
$$f_\varphi^{(k)} = \frac{1}{\sin\theta\partial\varphi}[(\nabla^2+2)w^{(k)}] - \tilde{L}_{21}^{(k)}(\omega_\theta^{(k)}) - \tilde{L}_{22}^{(k)}(\omega_\varphi^{(k)}) - \tilde{p}_k\omega_\varphi^{(k)} + K_{(k)}\delta_{(k)}\tilde{q}_2 = 0$$
$$f_z^{(k)} = (1+\nu^{(k)})\left[(\nabla^2+2)w^{(k)} - \frac{\partial\omega_\theta^{(k)}}{\partial\theta} - \operatorname{ctg}\theta\,\omega_1^{(k)} - \frac{\partial\omega_\varphi^{(k)}}{\sin\theta\partial\varphi}\right] +$$
$$+ C_{(k)}^2\left\{\left[\left(\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\right)\nabla^2 - \operatorname{ctg}^2\theta\frac{\partial}{\partial\theta}\right]\omega_\theta^{(k)} + \frac{\partial}{\sin\theta\partial\varphi}(\nabla^2 - 2\operatorname{ctg}\theta)\frac{\partial}{\partial\theta}\omega_\varphi^{(k)}\right\} +$$
$$+ \delta_{(k)}\varphi_{(k)}(w^{(1)} - w^{(2)}) + \tilde{p}_{(k)}\left(\frac{\partial\omega_\theta^{(k)}}{\partial\theta} + \operatorname{ctg}\theta\,\omega_\theta^{(k)} + \frac{\partial\omega_\varphi^{(k)}}{\sin\theta\partial\varphi}\right) -$$
$$- K_{(k)}H_{(k)}\left(\frac{\partial\tilde{q}_1}{\partial\theta} + \operatorname{ctg}\theta\,\tilde{q}_1 + \frac{\partial\tilde{q}_2}{\sin\theta\partial\varphi}\right) = 0 \tag{4.3}$$
$$\mu_1 = \frac{\omega_\theta^{(2)}}{\rho_{(2)}} - \frac{\omega_\theta^{(1)}}{\rho_{(1)}} + \frac{\partial}{\rho_{(1)}\rho_{(2)}\partial\theta}[\rho_{(2)}w^{(1)} - \rho_{(1)}w^{(2)} - h_0^*(w^{(1)} + w^{(2)})] +$$
$$+ \tilde{q}_1 - K\frac{\partial}{\partial\theta}\left(\frac{\partial\tilde{q}_1}{\partial\theta} + \tilde{q}_1\operatorname{ctg}\theta + \frac{\partial\tilde{q}_2}{\sin\theta\partial\varphi}\right) = 0$$
$$\mu_2 = \frac{\omega_\varphi^{(2)}}{\rho_{(2)}} - \frac{\omega_\varphi^{(1)}}{\rho_{(1)}} + \frac{\partial}{\rho_{(1)}\rho_{(2)}\sin\theta\partial\varphi}[\rho_{(2)}w^{(1)} - \rho_{(1)}w^{(2)} - h_0^*(w^{(1)} + w^{(2)})] +$$
$$+ \tilde{q}_2 - K\frac{\partial}{\sin\theta\partial\varphi}\left(\frac{\partial\tilde{q}_1}{\partial\theta} + \tilde{q}_1\operatorname{ctg}\theta + \frac{\partial\tilde{q}_2}{\sin\theta\partial\varphi}\right) = 0$$

Here, the notation for the operators

$$\tilde{L}_{11}^{(k)} = \nabla^2 - \nu^{(k)} - \operatorname{ctg}^2\theta - \frac{1+\nu^{(k)}}{2}\frac{\partial^2}{\sin^2\theta\partial\varphi^2}$$
$$\tilde{L}_{12}^{(k)} = \frac{\partial}{\sin\theta\partial\varphi}\left(\frac{1+\nu^{(k)}}{2}\frac{\partial}{\partial\theta} - \frac{3-\nu^{(k)}}{2}\operatorname{ctg}\theta\right) = \tilde{L}_{21}^{(k)}$$
$$\tilde{L}_{22}^{(k)} = \frac{1-\nu^{(k)}}{2}\frac{\partial}{\sin\theta\partial\varphi}(\nabla^2 + 1 - \operatorname{ctg}^2\theta) + \frac{1+\nu^{(k)}}{2}\frac{\partial^2}{\sin^2\theta\partial\varphi^2}$$

has been introduced. These operators are obtained from the operators (1.11) when conditions (4.1) are satisfied.

Instead of the required functions $\omega_\theta^{(k)}$, $\omega_\varphi^{(k)}$, $\tilde{q}_k$ (k = 1, 2), we introduce the new required unknowns $F^{(k)}$, $\Phi^{(k)}$, Q, Ψ in accordance with the representations

$$\omega_\theta^{(k)} = \frac{\partial F^{(k)}}{\partial\theta} + \frac{\partial\Phi^{(k)}}{\sin\theta\partial\varphi}, \quad \omega_\varphi^{(k)} = \frac{\partial F^{(k)}}{\sin\theta\partial\varphi} - \frac{\partial\Phi^{(k)}}{\partial\theta}$$
$$\tilde{q}_1 = \frac{\partial Q}{\partial\theta} + \frac{\partial\Psi}{\sin\theta\partial\varphi}, \quad \tilde{q}_2 = \frac{\partial Q}{\sin\theta\partial\varphi} - \frac{\partial\Psi}{\partial\theta} \tag{4.4}$$

After substituting expressions (4.4) into relations (4.3) and some reduction with an accuracy $O(C_{(k)}^2)$, we arrive at the equations

$$f_\theta^{(k)} = \frac{\partial}{\partial\theta}U^{(k)} + \frac{\partial}{\sin\theta\partial\varphi}V^{(k)} = 0, \quad f_\varphi^{(k)} = \frac{\partial}{\sin\theta\partial\varphi}U^{(k)} - \frac{\partial}{\partial\theta}V^{(k)} = 0$$

$$f_z^{(k)} = (\nabla^2+2)w^{(k)} - \left(\nabla^2 - \frac{C_{(k)}^2}{1+\nu^{(k)}}\nabla^2\nabla^2\right)F^{(k)} + \frac{\tilde{p}_{(k)}}{1+\nu^{(k)}}\nabla^2 F^{(k)} +$$
$$+ \frac{\varphi_{(k)}\delta_{(k)}}{1+\nu^{(k)}}(w^{(1)} - w^{(2)}) - \frac{K_{(k)}H_{(k)}}{1+\nu^{(k)}}\nabla^2 Q = 0 \tag{4.5}$$

$$\mu_1 = \frac{\partial M}{\partial\theta} + \frac{\partial N}{\sin\theta\partial\varphi} = 0, \quad \mu_2 = \frac{\partial M}{\sin\theta\partial\varphi} - \frac{\partial N}{\partial\theta} = 0$$

in which

$$U^{(k)} = (\nabla^2+2)w^{(k)} - (\nabla^2 + 1 - \nu^{(k)})F^{(k)} - \tilde{p}_{(k)}F^{(k)} + K_{(k)}\delta_{(k)}Q$$

$$V^{(k)} = \frac{1-\nu^{(k)}}{2}(\nabla^2+2)\Phi^{(k)} + \tilde{p}_{(k)}\Phi^{(k)} - K_{(k)}\delta_{(k)}\Psi$$

$$M = \frac{F^{(2)}}{\rho_{(2)}} - \frac{F^{(1)}}{\rho_{(1)}} + \frac{(\rho_{(2)} - h_0^*)w^{(1)} - (\rho_{(1)} + h_0^*)w^{(2)}}{\rho_{(1)}\rho_{(2)}} + (1 - K\nabla^2)Q \tag{4.6}$$

$$N = \frac{\Phi^{(2)}}{\rho_{(2)}} - \frac{\Phi^{(1)}}{\rho_{(1)}} + \Psi$$

If follows from Eqs (4.5) that the initial system of interconnected differential equations of stability (4.3) in the new unknowns decomposes into two independent systems of equations. Since, for all unknown functions in the case of a closed spherical shell, conditions for their periodicity with respect to the angular coordinate θ and φ are formulated instead of boundary conditions, those modes of loss of stability which are realized in a shell without the appearance of deflections of the load-bearing layers in the perturbed state are described by one of these systems of equations, which has the form

$$V^{(k)} = 0, \quad k = 1,2, \quad N = 0 \tag{4.7}$$

This follows from an analysis of the expressions for $V^{(k)}$ and N from relations (4.6). The modes of loss of stability, which are accompanied by deflections of the load-bearing layers in the perturbed state are described by the other system of equations

$$U^{(k)} = 0, \quad f_z^{(k)} = 0, \quad k = 1,2, \quad M = 0 \tag{4.8}$$

which contains the deflections $w^{(k)}$ and the scalar potential function $F^{(k)}$, Q. In the light of the results previously obtained in [1, 3], the modes of loss of stability described by Eqs (4.7) should be regarded as pure shear modes and the modes which are established by the solution of Eqs (4.8) should be regarded as mixed flexural modes.

Since the boundary conditions in spherical shell are replaced by conditions of periodicity of the functions $w^{(k)}$, $F^{(k)}$, $\Phi^{(k)}$, Q, Ψ with respect to the angular coordinates θ and φ, the following combinations are solutions of Eqs (4.7) and (4.8)

$$F = P_n^m(\cos\theta)(F_{nm}\cos m\varphi + \tilde{F}_{nm}\sin m\varphi) \tag{4.9}$$

where the function F is understood to be one of those shown above; F_{nm} and $\tilde{F}_{nm}$ are constants of integration, $P_n^m(\cos\theta)$ are associated Legendre polynomials of degree n, and m is the number of nodal meridians.

Here, the function F satisfies Legendre's equation

$$\frac{d^2F}{d\theta^2} + \mathrm{ctg}\,\theta\frac{dF}{d\theta} + \left(\lambda_n^2 - \frac{m^2}{\sin^2\theta}\right)F = 0, \quad \lambda_n^2 = n(n+1) \tag{4.10}$$

In accordance with properties (4.9) and (4.10), we can change from differential equations (4.7) and (4.8), to algebraic equations which have the same form both with respect to F_{nm} and with respect to $\tilde{F}_{nm}$. To do this, it is sufficient to carry out a formal transformation from $w^{(k)}$, $F^{(k)}$, $\Phi^{(k)}$, Q, Ψ to their amplitude values $w_{nm}^{(k)}$, $F_{nm}^{(k)}$, $\Phi_{nm}^{(k)}$, Q_{nm}, Ψ_{nm} or $\tilde{w}_{nm}^{(k)}$, $\tilde{F}_{nm}^{(k)}$, $\tilde{\Phi}_{nm}^{(k)}$, $\tilde{Q}_{nm}$, $\tilde{\Psi}_{nm}$ by replacing the operator ∇^2 by the numerical value $-\lambda_n^2$.

Shear modes of loss of stability. According to what has been discussed above for the investigation of shear modes of loss of stability, we have Eqs (4.7) which, when account is taken of expressions (4.6), can be represented in the algebraic form

$$\left[\frac{1-\nu^{(k)}}{2}(2-\lambda_n^2) + \tilde{p}^{(k)}\right]\Phi_{nm}^{(k)} - K_{(k)}\delta_{(k)}\Psi_{nm} = 0; \quad k = 1,2$$

$$\frac{\Phi_{nm}^{(2)}}{\rho_{(2)}} - \frac{\Phi_{nm}^{(1)}}{\rho_{(1)}} + \Psi_{nm} = 0$$

Note that analogous equations are also obtained in the amplitude values $\tilde{\Phi}_{nm}^{(k)}$, $\tilde{\Psi}_{nm}$.

From the condition for the solutions of these system to be non-trivial, we arrive at an equation for determining the critical external pressure p

$$\tilde{p}_{(1)}\tilde{p}_{(2)} + \tilde{p}_{(1)}\left[\frac{1-\nu^{(2)}}{2}(2-\lambda_n^2) - \frac{K_{(2)}}{\rho_{(2)}}\right] + \tilde{p}_{(2)}\left[\frac{1-\nu^{(1)}}{2}(2-\lambda_n^2) - \frac{K_{(1)}}{\rho_{(1)}}\right] +$$
$$+ \frac{(1-\nu^{(1)})(1-\nu^{(2)})}{4}(2-\lambda_n^2)^2 - \left[\frac{(1-\nu^{(2)})K_{(1)}}{2\rho_{(1)}} + \frac{(1-\nu^{(1)})K_{(2)}}{2\rho_{(2)}}\right](2-\lambda_n^2) = 0 \tag{4.11}$$

An investigation of Eq. (4.11) together with (4.2) shows that the values of p are obtained as negative quantities when $n = 0$ ($\lambda_n = 0$) and are increasing positive quantities as the order of the associated functions $P_n^m(\cos\theta)$ increases. The pressure p attains its smallest value, denoted by p^c, when $n = 1$ and its magnitude is found from the relation

$$p^c = \frac{2B^{(2)}(1+\chi_{(1)}^* + \chi_{(2)}^*)}{(1+\chi_{(1)}^*)}\left[\frac{K_{(1)}}{\rho_{(1)}} + \frac{(1+\chi_{(1)}^*)(1+\nu^{(2)})}{\chi_{(1)}^*(1+\nu^{1)})}\frac{K_{(2)}}{\rho_{(2)}}\right] \tag{4.12}$$

At the same time, loss of stability is possible when $m = 0$ ($P_1^0(\cos\theta) = \cos\theta$) or when $m = 1$ ($P_1^1(\cos\theta) = -\sin\theta$).

In the first case, in accordance with representations (4.4) and (4.9), we have

$$P_n^m(\cos\theta)\Phi_{nm}^{(k)} = \Phi_{10}^{(k)}\cos\theta, \quad P_n^m(\cos\theta)\Psi_{nm} = \Psi_{10}\cos\theta \tag{4.13}$$

since there are no amplitude quantities $\tilde{\Phi}_{10}^{(k)}$ and $\tilde{\Psi}_{10}$ when $m = 0$. In this case, the complete system of differential equations (4.5) will be satisfied when

$$w^{(k)} = 0, \quad F^{(k)} = 0, \quad Q = 0 \tag{4.14}$$

Hence, here in the perturbed state, on the basis of relations (3.17), (4.4), (4.13) and (4.14), the displacements $w^{(k)}$, $u^{(k)}$ and $\upsilon^{(k)}$ and the functions $\tilde{q}_1$ and q_2 will have the form

$$w^{(k)} = 0, \quad u^{(k)} = 0, \quad \tilde{q}_1 = 0, \quad \upsilon^{(k)} = \Phi_{10}^{(k)}\sin\theta, \quad \tilde{q}_2 = \Psi_{10}\sin\theta \tag{4.15}$$

The mode of loss of stability has been investigated in detail in [3] for a shell with a thin filler. However, according to the recent results in [6], the value of critical load obtained needs to be improved. This mode of loss of stability occurs by mutual rotation of one of the load-bearing layers with respect to the other, with the axis of rotation passing through the centre of the sphere. As the shell transfers from the unperturbed state (that is, from the solution (2.6), (2.8), (2.9)) into the perturbed state (that is, into the solution (4.15)), this rotation occurs at the initial stage (that is, on infinitesimal initial section of the new trajectory) without deformations and bending of the load-bearing layers.

In addition to the mode of loss of stability (4.13), (4.15), another mode (the case when $m = 1$) exists, which is described by the functions

$$\Phi^{(k)} = (\Phi_{11}^{(k)}\cos\varphi + \tilde{\Phi}_{11}^{(k)}\sin\varphi)\sin\theta, \quad \Psi = -(\Psi_{11}\cos\varphi + \tilde{\Psi}_{11}\sin\varphi)\sin\theta \tag{4.16}$$

In this case, instead of (4.15), we will have

$$\begin{aligned} &w^{(k)} = 0, \quad u^{(k)} = -\Phi_{11}^{(k)}\sin\varphi + \tilde{\Phi}_{11}^{(k)}\cos\varphi \\ &\upsilon^{(k)} = (\Phi_{11}^{(k)}\cos\varphi + \tilde{\Phi}_{11}^{(k)}\sin\varphi)\cos\theta \\ &\tilde{q}_1 = -\Psi_{11}\sin\varphi + \tilde{\Psi}_{11}\cos\varphi, \quad \tilde{q}_2 = (\Psi_{11}\cos\varphi + \tilde{\Psi}_{11}\sin\varphi)\cos\theta \end{aligned} \tag{4.17}$$

and, as previously, the value of the critical load is found from Eq. (4.12) using formula (4.2) for K_i, which are independent of the initial compression of the filler $\tilde{q}$.

In the case of a shell of symmetrical structure with a thin-walled filler

$$(\rho_{(1)} \approx \rho_{(2)} \approx 1, \nu_1^{(1)} = \nu^{(2)} = \nu, E^{(1)} = E^{(2)} = E, \overset{0}{t}_{(1)} = \overset{0}{t}_{(2)} = t_0)$$

and the formula

$$p^c = \frac{G_3(1+2\chi)}{\chi\{1 + h_0(1+\chi)/[4(1+2\chi)\chi]\}}, \quad \chi = \frac{E_3(1-\nu)}{8h_0t_0E} \tag{4.18}$$

follows from relation (4.12). By virtue of the fact that the non-linear terms are retained in relations (1.1), this formula fundamentally improves the analogous formula obtained in [3], which only uses the linear equations of the theory of elasticity for the filler [8, 9].

Mixed flexural mode of loss of stability. Flexural modes of loss of stability are described by the system of equations (4.8) which, according to relations (4.9) and (4.10), take the form

$$\begin{aligned} &(2-\lambda_n^2)W_{nm}^{(k)} + (\lambda_n^2 - 1 + \nu^{(k)} - \tilde{p}_{(k)})F_{nm}^{(k)} + K_{(k)}\delta_{(k)}Q_{nm} = 0 \\ &(2-\lambda_n^2)W_{nm}^{(k)} + \lambda_n^2\left(1 + \frac{C_{(k)}^2\lambda_n^2 - \tilde{p}_{(k)}}{1+\nu^{(k)}}\right)F_{nm}^{(k)} + \frac{\varphi_{(k)}\delta_{(k)}}{1+\nu^{(k)}}(W_{nm}^{(1)} - W_{nm}^{(2)}) + \\ &+\frac{K_{(k)}H_{(k)}}{1+\nu^{(k)}}\lambda_n^2 Q_{nm} = 0; \quad k = 1,2 \\ &\frac{\rho_{(2)}W_{nm}^{(1)} - \rho_{(1)}W_{nm}^{(2)} - h_0^*(W_{nm}^{(1)} + W_{nm}^{(2)})}{\rho_{(1)}\rho_{(2)}} + \frac{F_{nm}^{(2)}}{\rho_{(2)}} - \frac{F_{nm}^{(1)}}{\rho_{(1)}} + (1 + K\lambda_n^2)Q_{nm} = 0 \end{aligned} \tag{4.19}$$

An analogous system of algebraic equations is also obtained for the amplitudes $\tilde{W}_{nm}^{(k)}$, $\tilde{F}_{nm}^{(k)}$ and Q_{nm}.

Eliminating the amplitudes $\tilde{W}_{nm}^{(k)}$ ($k = 1, 2$) and $\tilde{Q}_{nk}$, from system (4.19), we arrive at the following algebraic equations

$$\begin{aligned}(A_n^{(1)} - a_n^{(1)}\tilde{P}_{(1)})F_{nm}^{(1)} - (B_n^{(1)} - b_n^{(1)}\tilde{P}_{(2)})F_{nm}^{(2)} &= 0\\ (B_n^{(2)} + b_n^{(2)}\tilde{P}_{(1)})F_{nm}^{(1)} + (A_n^{(2)} - a_n^{(2)}\tilde{P}_{(2)})F_{nm}^{(2)} &= 0\end{aligned} \tag{4.20}$$

where

$$A_n^{(k)} = 1 - \nu^{(k)2} + C_{(k)}^2\lambda_n^4 + \varphi_{(k)}\left(1 + \frac{\nu^{(k)}}{\lambda_n^2 - 2}\right) - K_n^{(k)} f_{(k)}$$

$$B_n^{(k)} = \varphi_{(k)}\left(1 + \frac{1 + \nu^{(3-k)}}{\lambda_n^2 - 2}\right) + K_n^{(k)} f_{(3-k)}$$

$$a_n^{(k)} = \lambda_n^2 - 1 - \nu^{(k)} + \frac{\varphi_{(k)}}{\lambda_n^2 - 2} - K_n^{(k)}(\delta_{(k)} + h_0 - h_0^*)$$

$$b_n^{(k)} = \frac{\varphi_{(k)}}{\lambda_n^2 - 2} - K_n^{(k)}(\delta_{(k)} - h_0 + h_0^*)$$

$$f_{(k)} = (1 + \nu^{(k)})\delta_{(k)} + (h_0 - h_0^*)(1 + \nu^{(k)}) - h_0^*(\lambda_n^2 - 2)$$

$$K_n^{(k)} = \frac{K_{(k)}H_{(k)}\lambda_n^2 - K_{(k)}(1 + \nu^{(k)})\delta_{(k)} + \varphi_{(k)}(K_{(1)} + K_{(2)})\delta_{(k)}/(\lambda_n^2 - 2)}{\rho_{(1)}\rho_{(2)}(\lambda_n^2 - 2)(1 + K\lambda_n^2) + K_{(1)}(\rho_{(2)} - h_0^*) + K_{(2)}(\rho_{(1)} - h_0^*)}$$

Introducing the dimensionless load parameter

$$\bar{q} = p(1 + \chi_{(1)}^*)/[2(1 + \chi_{(1)}^* + \chi_{(2)}^*)B^{(2)}] \tag{4.21}$$

in accordance with which

$$\tilde{P}_{(1)} = \chi_{(1)}^*(1 + \nu^{(1)})\bar{q}/[(1 + \chi_{(1)}^*)(1 + \nu^{(2)})], \quad \tilde{P}_{(2)} = \bar{q}$$

we obtain, from system (4.20), a quadratic equation in $\bar{q}$. Its solution can be represented in the form

$$\bar{q}_{1,2} = A_n \pm \sqrt{A_n^2 - B_n} \tag{4.22}$$

where

$$A_n = \frac{1}{2(a_n^{(1)}a_n^{(2)} - b_n^{(1)}b_n^{(2)})}\left[a_n^{(1)}A_n^{(2)} - b_n^{(2)}B_n^{(1)} + \frac{(1 + \chi_{(1)}^*)(1 + \nu^{(2)})}{\chi_{(1)}^*(1 + \nu^{(1)})}(a_n^{(2)}A_n^{(1)} - b_n^{(1)}B_n^{(2)})\right]$$

$$B_n = \frac{(1 + \chi_{(1)}^*)(1 + \nu^{(2)})(A_n^{(1)}A_n^{(2)} - B_n^{(1)}B_n^{(2)})}{\chi_{(1)}^*(1 + \nu^{(1)})(a_n^{(1)}a_n^{(2)} - b_n^{(1)}b_n^{(2)})}$$

We will denote the minimum positive value of the roots of (4.22) by $\bar{q}_*$. It is found from the solution of the problem of minimizing the functional

$$\bar{q}_* = \min_{(n)}(\bar{q}_1, \bar{q}_2) \tag{4.23}$$

If n_* is the value of n for which a minimum of the load parameter $\bar{q}_*$ is obtained then the modes of loss of stability will be described by an associated Legendre function of degree $n_*(P^m_{n*}(\cos\theta))$ with a number of nodal meridians m which varies in the range $0 \le m \le n_*$.

In this case, the mode of loss of stability will be described by the functions

$$
\begin{aligned}
w^{(k)} &= P_n^{\tilde{m}}(\cos\theta)(W_{n\tilde{m}}^{(k)}\cos\tilde{m}\varphi + \tilde{W}_{n\tilde{m}}^{(k)}\sin\tilde{m}\varphi) \\
u^{(k)} &= -\frac{d}{d\theta}[P_n^{\tilde{m}}(\cos\theta)](F_{n\tilde{m}}^{(k)}\cos\tilde{m}\varphi + \tilde{F}_{n\tilde{m}}^{(k)}\sin\tilde{m}\varphi) \\
v^{(k)} &= \frac{\tilde{m}P_n^{\tilde{m}}(\cos\theta)}{\sin\theta}(F_{n\tilde{m}}^{(k)}\sin\tilde{m}\varphi - \tilde{F}_{n\tilde{m}}^{(k)}\cos\tilde{m}\varphi) \\
q_1 &= -\frac{d}{d\theta}[P_n^{\tilde{m}}(\cos\theta)](Q_{n\tilde{m}}\cos\tilde{m}\varphi + \tilde{Q}_{n\tilde{m}}\sin\tilde{m}\varphi) \\
q_2 &= \frac{\tilde{m}P_n^{\tilde{m}}(\cos\theta)}{\sin\theta}(Q_{n\tilde{m}}\sin\tilde{m}\varphi - \tilde{Q}_{n\tilde{m}}\cos\tilde{m}\varphi)
\end{aligned}
$$

where $0 \le \tilde{m} \le n_*$.

5. NUMERICAL RESULTS AND THEIR ANALYSIS

The critical loads and the modes of loss of stability corresponding to them were investigated for shells of symmetrical structure, when

$$\nu^{(k)} = \nu, \quad E_2^{(k)} = E, \quad \overset{0}{t}_{(k)} = t_0$$

and, consequently,

$$\chi_{(k)}^* = \chi = \frac{E_3\rho_{(1)}\rho_{(2)}(1-\nu)}{8h_0t_0E}, \quad \varphi_{(k)} = \varphi = \frac{E_3\rho_{(1)}\rho_{(2)}(1-\nu^2)}{4h_0t_0E}, \quad C_{(k)}^2 = \frac{t_0^2}{3\rho_{(k)}^2}$$

$$K_{(k)} = \frac{3G_3^*\rho_{(1)}^2\rho_{(2)}^2(1-\nu^2)}{4h_0t_0(3+h_0^2)E\rho_{(k)}}; \quad k = 1,2$$

The numerical results were obtained by varying the parameters t_0, χ, $\delta = G_3/E_3$, $r = h_0/t_0$ within the following limits

$$0.01 \le t_0 \le 0.0001, \quad 0.1 \le \chi \le 5, \quad 1/2.6 \le \delta \le 0.01/2.6, \quad 2 \le r \le 10 \tag{5.1}$$

Some of these results for $\nu = 0.3$ and $t_0 = 0.01$ are shown in Table 1 when $\chi = 1$, and in Table 2, in which the following notation is introduced: $m_*^u = p_*^u/p_*$, $m_*^c = p_*^c/p_*$, p_*^u is the critical pressure in the case of a flexural mode found from the solution of problem (4.23) when $q_0^3 \ne 0$ (the improved solution constructed in this paper) and p_* is the critical pressure for a single-layer spherical shell which, in the notation adopted in this paper, is calculated using the formula

$$p_* = \frac{8Et_0^2}{\rho_{(2)}^2\sqrt{3(1-\nu^2)}} = \frac{4t_0B^{(2)}}{\rho_{(2)}}\sqrt{\frac{1-\nu^2}{3}}$$

An analysis of the results obtained shows that the tendency for loss of stability by a shear mode becomes stronger earlier than for loss of stability by a flexural mode, in particular, when the transverse shear modulus of the filler is reduce compared with the modulus E_3 (that is, the anisotropy coefficient $\delta = G_3/(2.6E_3)$ is reduced; $\delta = 1/2.6$ corresponds to an isotropic filler) and, also, when h_0 is increased and χ is reduced. Since the value of p_*^c is directly proportional to the parameter δ, m_*^c becomes smaller than m_*^u only for small values of δ while the value of m_*^u is practically independent of δ. As can be seen from Table 2, $p_*^u < p_*^c$ everywhere in the case of a shell with an isotropic filler. The difference between the values of the critical loads for a spherical three-layer shell and a single-layer shell (that is, an isolated external load-bearing layer) is not so significant (Table 2) as, for example, in the case of three-layer plates. As would be expected, this difference becomes larger when t_0 is reduced and the parameters χ and r are increased.

Table 1

r	$\delta = 1/2.6$	0.1/2.6	0.01/2.6
	m_*^u	m_*^u	m_*^u
2	2.05	1.98	1.95
4	2.31	2.03	2.00
6	2.71	2.10	2.04
8	3.26	2.19	2.08
10	3.98	2.29	2.12

Table 2

χ	r	$\delta = 1/2.6$		0.1/2.6	0.01/2.6
		m_*^u	m_*^c	m_*^u	m_*^c
0.2	2	1.42	2.599	1.42	1.42
	10	1.64	14.006	1.45	1.43
0.4	2	1.70	3.347	1.68	1.68
	10	2.23	18.231	1.73	1.73
0.6	2	1.87	4.096	1.84	1.84
	10	2.81	22.450	1.93	1.93
0.8	2	1.98	4.844	1.93	1.92
	10	3.38	26.666	2.18	2.05
1	2	2.05	5.592	1.98	1.97
	10	3.92	30.879	2.29	2.12
5	2	2.47	20.558	2.10	2.06
	10	12.11	115.070	3.26	2.34

Calculations were also carried out to determine the critical pressure in the case of a flexural mode of loss of stability using formula (4.23) when $q_3^0 = 0$ (the solution which corresponds to the formulation described in [3]). Comparison with the values of m_*^u showed that the improved versions of the theory of three-layer shells constructed earlier in [7, 8], under the assumption that the transverse shear deformations are small when linear kinematic relations are used to determine the transverse compression deformation of the filler, on the basis of which the investigation of flexural modes of loss of stability was carried out in [2, 3], are extremely accurate in describing these modes of loss of stability. Consequently, the use of non-linear kinematic relations, in the case of the filler, to determine the deformation ε_{zz} in the unperturbed state is only necessary for a correct description of the shear modes of loss of stability for three-layer structures.

Attention is drawn to the fact that, in the case of a small value of the transverse compression parameter χ, the value of m_*^u is even less than two (that is, only the upper load-bearing layer, which is strengthened by the filler, loses stability in a mixed flexural mode).

The effect of having three-layers (that is, $m_*^u > 2$) in the case of a shell is only attained for certain combinations of the parameters χ and r, and becomes more pronounced as these parameters increase. It is manifests itself to the greatest extent in the case of shells with an isotropic filler ($\delta = 1/2.6$).

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